

POSITIVELY CURVED MANIFOLDS WITH MAXIMAL SYMMETRY-RANK

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ABSTRACT. The symmetry-rank of a riemannian manifold is by definition the rank of its isometry group. We determine precisely which smooth closed manifolds admit a positively curved metric with maximal symmetry-rank.

INTRODUCTION

Due to obvious ambiguities there are many interesting aspects to the following general type problem: *Classify positively curved riemannian manifolds with large isometry groups.* One example of this is of course the well-known classification of homogeneous manifolds with positive (sectional) curvature (cf. [2, 3, 4, 17]). In this paper, “large” refers to the rank of the isometry group. Specifically, we define the symmetry rank of a riemannian manifold M to be

$$\text{symrank}(M) = \text{rank Iso}(M),$$

where $\text{Iso}(M)$ is the isometry group of M . The following solution to the above problem in this context was inspired by the work of Allday and Halperin in [1] and of Hsiang and Kleiner in [11].

Theorem. *Let M be an n -dimensional, closed, connected riemannian manifold with positive sectional curvature. Then*

- (i) $\text{symrank}(M) \leq \lfloor (n+1)/2 \rfloor$.

Moreover,

- (ii) *Equality holds in (i) only if M is diffeomorphic to either a sphere, a real or complex projective space, or a lens space.*

Note that any of the manifolds listed in (ii) above with their standard metrics have maximal symmetry rank (cf. Example 2.3). The main techniques used in the proof of this result are convexity and critical point theory (cf. [9]) applied to the positively curved, singular, orbit spaces. For other contributions to the general problem we refer to [10, 15] and [16]. It is our pleasure to thank Stephan Stolz for suggestions leading to diffeomorphism classification rather than topological classification.

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1. CIRCLE ACTIONS WITH MAXIMAL FIXED POINT SET

In this section we will show how the presence of an effective isometric circle action with large fixed-point set determines the type of the manifold. First let us recall, that the fixed-point set $\text{Fix}(S^1)$ of an isometric S^1 -action is a union of totally geodesic submanifolds of even codimension. Here we let $\dim \text{Fix}(S^1)$ denote the largest dimension of components of $\text{Fix}(S^1)$. Note also, that the above statements hold as well for $\text{Fix}(\mathbb{Z}_q)$, where \mathbb{Z}_q , is any subgroup of S^1 . From now on we confine our attention to positively curved manifolds M . The key observation is already contained in the following.

Lemma 1.1. *Suppose $\text{codim } \text{Fix}(S^1) = 2$. Then:*

- (i) *Exactly one component, N , of $\text{Fix}(S^1)$ has codimension 2.*
- (ii) *There is a unique orbit $S^1 p_0$ at maximal distance from N .*
- (iii) *S^1 acts freely on $M - (N \cup S^1 p_0)$.*

Proof. Since any two totally geodesic submanifolds N_1 and N_2 with $\dim N_1 + \dim N_2 \geq \dim M$ intersect by a Synge type argument, part (i) is clear.

Now consider the orbit space $X = M/S^1$. This is a positively curved space (in Aleksandrov distance comparison sense, cf. [5]), with totally geodesic boundary $\partial X = N$. As in riemannian geometry (cf. [6]), $\text{dist}(N, -) : M/S^1 \rightarrow \mathbb{R}$ is a strictly concave function (for the case of general Aleksandrov spaces see [14]). In particular, there is a unique point $\bar{p}_0 \in M/S^1$ (the soul) at maximal distance from N . This proves (ii).

To prove (iii), consider a point $\bar{p} \in X - (N \cup \bar{p}_0)$. Since \bar{p} is in the boundary of the convex set $\{\bar{x} \in X \mid \text{dist}(N, \bar{x}) \geq \text{dist}(N, \bar{p})\}$, every segment (minimal geodesic) from \bar{p} to N makes an angle $> \pi/2$ to every segment from \bar{p} to \bar{p}_0 . If $p \in M$ and $S^1 p = \bar{p}$, the same statement holds for horizontal lifts of segments to p . Clearly, the isotropy S_p^1 at p will preserve each set of “opposite” segments. Consequently, by the angle condition above, S_p^1 will fix a direction pointing toward N (i.e. making an angle $< \pi/2$ to all segments from p to N and an angle $> \pi/2$ to all segments from p to $S^1 \cdot p_0$). The same argument, however, also applies to a point $q \in \text{Fix}(S_p^1)$ closest to N . Since all points sufficiently close to N have trivial isotropy, this leads to a contradiction unless S_p^1 is trivial. \square

The structure obtained in Lemma 1.1 and its proof, provides the ingredients for establishing our main recognition result.

Theorem 1.2. *Let M be a closed, connected riemannian manifold with positive curvature. If M admits an effective isometric S^1 -action with $\text{codim } \text{Fix}(S^1) = 2$, then M is diffeomorphic to either the unit sphere S_1^n , a space form S_1^n/\mathbb{Z}_q , or $\mathbb{C}P^m = S_1^{n+1}/S^1$ (when $n = 2m$).*

Proof. Let p_0, \bar{p}_0 and N be as in Lemma 1.1. When N is viewed as a subset of M/S^1 , we denote it by \bar{N} . From the angle condition obtained in Lemma 1.1 it follows that $\text{dist}(\bar{N}, -)$ and $\text{dist}(\bar{p}_0, -)$ (resp. $\text{dist}(N, -)$ and $\text{dist}(S^1 \cdot p_0, -)$) have no critical points in $M/S^1 - (\bar{N} \cup \bar{p}_0)$ (resp. $M - (N \cup S^1 \cdot p_0)$). Choose gradient-like vector fields, \bar{V} and V on M/S^1 and M respectively, so that V is a

horizontal lift of \bar{V} on $M - (N \cup S^1 \cdot p_0)$ which is radial near N and $S^1 \cdot p_0$ (cf. e.g. [9]). This shows in particular that M is the union of tubular neighborhoods $D_\varepsilon(N)$ and $D_\varepsilon(S^1 \cdot p_0)$ of N and $S^1 \cdot p_0$, respectively. Moreover, $N \cong \bar{N}$ is diffeomorphic to $\partial D_\varepsilon(\bar{p}_0) \cong \partial D_\varepsilon(S^1 p_0)/S^1 \simeq \partial D_\varepsilon^\perp(p_0)/S_{p_0}^1$, where $D_\varepsilon^\perp(p_0)$ denotes the ε -normal disc at p_0 to $S^1 \cdot p_0$. In view of this, it is not surprising that M is determined by the isotropy group $S_{p_0}^1$ at p_0 .

Case 1: $S_{p_0}^1 = \{1\}$. Clearly $(M/S^1, \bar{N})$ is diffeomorphic to (D^{n-1}, S^{n-2}) and the quotient map $\pi : M - N \rightarrow M/S^1 - \bar{N} \simeq D^{n-1} - S^{n-2}$ is a (trivial) principal circle bundle (cf. Lemma 1.1 (iii)). Moreover, when restricted to $\partial D_\varepsilon(\bar{N}) \subset M$, $\pi : \partial D_\varepsilon(N) \rightarrow \partial D_\varepsilon(\bar{N}) \simeq \bar{N} \cong N(\simeq S^{n-2})$ is nothing but the normal projection in M . In particular, $D_\varepsilon(N) \simeq \partial D_\varepsilon(N) \times_{S^1} D^2$ and hence M is diffeomorphic to $(M - \text{int } D_\varepsilon(N)) \cup \partial D_\varepsilon(N) \times_{S^1} D^2 \cong D^{n-1} \times S^1 \cup S^{n-2} \times D^2 \simeq S^n$.

Case 2: $S_{p_0}^1 = S^1$. Note by Lemma 1.1 (iii) that $S_{p_0}^1$ acts freely on the unit sphere $S^{n-1} \simeq \partial D_\varepsilon(p_0)$ at p_0 . In particular, $n = 2m$ and $N \cong \bar{N} \simeq \mathbb{C}P^{m-1}$. As in the case above, $\pi : \partial D_\varepsilon(N) \rightarrow \partial D_\varepsilon(\bar{N}) \simeq \bar{N} \cong N(\simeq \mathbb{C}P^{m-1})$ is the normal projection in M and $D_\varepsilon(N)$ is diffeomorphic to $\partial D_\varepsilon(N) \times_{S^1} D^2$. Since $M - \text{int } D_\varepsilon(N) \simeq D_\varepsilon(p_0) \simeq D^{2m}$ and $S^{2m-1} \simeq \partial D_\varepsilon(N) \rightarrow N \simeq \mathbb{C}P^{m-1}$ is the Hopf map we conclude that M is diffeomorphic to $D^{2m} \cup S^{2m-1} \times_{S^1} D^2 \simeq \mathbb{C}P^m$.

Case 3: $S_{p_0}^1 = \mathbb{Z}_q$. First observe that $S_{p_0}^1$ acts freely on the unit normal sphere $S^{n-2} \simeq \partial D_\varepsilon^\perp(p_0)$, and consequently $N \cong \bar{N} \simeq S^{n-2}/\mathbb{Z}_q$. By proceeding as above or considering the q -fold universal cover $\tilde{M} \simeq S^n$ we find that M is diffeomorphic to $\mathbb{R}P^n$ ($q = 2$) or to a lens space $L_q \simeq S^n/\mathbb{Z}_q$ ($q > 2$ and n odd). \square

The main issue in Lemma 1.1 and Theorem 1.2 is that if $\text{codim } \text{Fix}(S^1) = 2$, then S^1 acts transitively on the fibers of the unit normal bundle of the maximal component $N \subset \text{Fix}(S^1)$. A generalization for a general group G with this property will be explored in [10]. The corresponding recognition is coarser and contains more topological types.

2. MAXIMAL TORUS ACTIONS

The purpose of this section is to show that any sufficiently large effective and isometric torus action on a positively curved manifold contains a circle action with large fixed-point set. This together with the results obtained in Section 1 will establish the Main Theorem in the Introduction.

We consider even and odd dimensions separately.

Theorem 2.1. *Let M be a closed, connected and positively curved riemannian manifold of dimension $2n$. Then any effective and isometric T^k -action has $k \leq n$. If $k = n$, T^n contains an S^1 with $\text{codim } \text{Fix}(S^1) = 2$.*

Proof. The proof is by induction on n . It is a simple and well-known fact that any isometric S^1 -action on an even-dimensional manifold with positive curvature has non-empty fixed-point set (cf. e.g. [12]). In dimension two, any such fixed point is of course isolated. To complete the induction anchor, suppose $T^2 = S^1 \times S^1$ acts on M^2 . If p is fixed by say the first circle factor, it must be fixed also by the second

factor. The induced T^2 -action on the unit circle at p must have S^1 isotropy. This S^1 will then act trivially on M^2 since its fixed-point set has even codimension. Thus, T^2 cannot act effectively on M .

Assume by induction that the claim has been established for $n \leq l$, and consider a T^{l+1} -action on $M^{2(l+1)}$. Pick an $S^1 \subset T^{l+1}$ with $\text{Fix}(S^1)$ of minimal codimension. We claim that $\text{codim } \text{Fix}(S^1) = 2$. If not, $\text{codim } N = 2(m+l)$, $m \geq 1$ for any component $N \subset \text{Fix}(S^1)$. Now, clearly $T^l = T^{l+1}/S^1$ acts on $N^{2(l-m)}$. By the induction hypothesis $T^l \times N \rightarrow N$ is ineffective, in fact some $T^m \subset T^l$ acts trivially on N . If $\pi : T^{l+1} \rightarrow T^{l+1}/S^1$ is the quotient map, this implies that $T^{m+1} = \pi^{-1}(T^m) \subset T^{l+1}$ fixes all points of N . Since $\text{codim } N = \text{codim } \text{Fix}(S^1)$ was assumed minimal, the induced action by T^{m+1} on any unit normal sphere, S^{2m+1} , to N is almost free. This, however, is possible only when $m = 0$ (cf. [8]).

To prove maximality suppose T^{l+2} acts isometrically on $M^{2(l+1)}$. From the above, some $S^1 \subset T^{l+2}$ has a fixed-point component N of dimension $2l$, and $T^{l+1} = T^{l+2}/S^1$ acts on N^{2l} . By the induction hypothesis some $S^1 \subset T^{l+2}/S^1$ acts trivially on N , and so does $T^2 = \pi^{-1}(S^1) \subset T^{l+1}$. As before, this implies that some $S^1 \subset T^2 \subset T^{l+2}$ acts trivially on M . □

The main difference between even and odd dimensions is related to the existence, respectively possible non-existence of fixed points for circle actions. Nonetheless, we have the following analogue to Theorem 2.1.

Theorem 2.2. *Let M be a closed, connected and positively curved riemannian manifold of dimension $2n+1$. Then any effective and isometric T^{k+1} -action has $k \leq n$. If $k = n$, T^{n+1} contains an S^1 with $\text{codim } \text{Fix}(S^1) = 2$.*

Proof. First consider the case $n = 1$. We must show that any T^2 action on M^3 has S^1 -isotropy. Note that if some isotropy group T_p^2 is non-trivial, the whole torus T^2 will act on the individual components of $\text{Fix } T_p^2$. As we have seen, this yields S^1 -isotropy. On the other hand, a free isometric $S^1 \times S^1$ -action on M^3 would induce a free isometric S^1 -action on the positively curved 2-manifold, $M^2 = M^3/S^1$ by O'Neill's formula (cf. [13] or [7]). This is impossible by Theorem 2.1. Using this fact as before, we see that any T^3 action on M^3 must be at least S^1 -ineffective.

By induction assume that Theorem 2.2 holds for $n \leq l$, and consider an isometric action $T^{l+2} \times M^{2l+3} \rightarrow M^{2l+3}$. If some $S^1 \subset T^{l+2}$ has non-empty fixed-point set we proceed exactly as in the proof of Theorem 2.1. As in the induction anchor, this is indeed the case unless T^{l+2} acts freely on M^{2l+3} . Such an action would induce a free T^{l+1} -action on a positively curved manifold $M^{2l+2} = M^{2l+3}/S^1$, which is impossible by Theorem 2.1. The maximality statement is now proved exactly as in Theorem 2.1. □

We conclude by exhibiting maximal torus-actions on the possible model spaces.

Example 2.3. Consider the standard action $T^{n+1} \times S^{2n+1} \rightarrow S^{2n+1}$ given in complex coordinates by $(e^{2\pi i \theta_0}, \dots, e^{2\pi i \theta_n}) \cdot (z_0, \dots, z_n) = (e^{2\pi i \theta_0} z_0, \dots, e^{2\pi i \theta_n} z_n)$. This also induces a maximal effective T^{n+1} -action on all lens spaces S^{2n+1}/\mathbb{Z}_q ,

as well as an effective $T^n = T^{n+1}/\Delta(S^1)$ -action on $\mathbb{C}P^n = S^{2n+1}/\Delta(S^1)$. To get a maximal effective action of T^n on S^{2n} simply suspend the above action $T^n \times S^{2n-1} \rightarrow S^{2n-1}$.

We finally point out that our Main Theorem in dimension 4 is not as general as the one by Hsiang and Kleiner [11] (they assume only S^1 -symmetry whereas we assume T^2 -symmetry). On the other hand our conclusion is stronger (by giving a diffeomorphism classification rather than a topological classification). It may also be worth mentioning that an analogous result for $SU(2)^n$ -actions (where one would add only $\mathbb{H}P^n$ to the list) does not hold, since e.g. $SU(2)^3$ acts on the 12-dimensional Flag manifold M^{12} [17] of quaternion lines in planes on \mathbb{H}^3 (cf. [15] and [10] though).

REFERENCES

- [1] C. Allday and S. Halperin, *Lie group actions on spaces of finite rank*, Quart. J. Math. Oxford 29 (1978) 63–67.
- [2] S. Aloff and N.L. Wallach, *An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures*, Bull. Amer. Math. Soc. 81 (1975) 93–97.
- [3] L. Berard Bergery, *Les variétés Riemanniennes homogènes simplement connexes de dimension impair à courbure strictement positive*, J. Math. Pures Appl. 55 (1976) 47–68.
- [4] M. Berger, *Les Variétés Riemanniennes homogènes normales simplement connexes à courbure strictement positive*, Ann. Scuola Norm. Sup. Pisa 15 (1961) 179–246.
- [5] Y. Burago, M. Gromov and G. Perelman, *Alexandrovs spaces with curvatures bounded from below I*, Uspeki Mat. Nauk 47 (1992) 3–51.
- [6] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of non-negative curvature*, Ann. of Math. 96 (1972) 413–443.
- [7] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. 16 (1967) 715–737.
- [8] D. Gromoll and K. Grove, *The low-dimensional metric foliations of Euclidean spheres*, J. Differential 28 (1988) 143–156.
- [9] K. Grove, *Critical point theory for distance functions*, in: Differential Geometry, Part 3, Proceedings of Symposia in Pure Mathematics, Vol. 54 (American Mathematical Society, Providence, RI, 1993) 357–385.
- [10] K. Grove and C. Searle, in preparation.
- [11] W.-Y. Hsiang and B. Kleiner, *On the topology of positively curved 4-manifolds with symmetry*, J. Differential Geom. 30 (1989) 615–621.
- [12] S. Kobayashi, *Transformation Groups in Differential Geometry*, Ergebnisse der Mathematik und Ihre Grenzgebiete, Band 70 (Springer, Berlin, 1972).
- [13] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. 13 (1966) 459–469.
- [14] G. Perelman, *Alexandrovs spaces with curvatures bounded from below II*, Preprint.
- [15] C. Searle, *Manifolds of positive curvature with large symmetries*, Ph.D. Thesis, University of Maryland at College Park, 1992.
- [16] C. Searle, *Cohomogeneity and positive curvature in low dimensions*, Math. Z., to appear.
- [17] N.R. Wallach, *Compact homogeneous manifolds with strictly positive curvature*, Ann. of Math. 96 (1972) 277–295.

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